The canonical complex of the weak order

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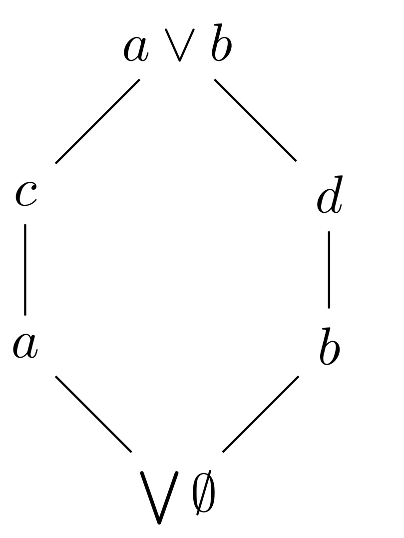
Canonical representations of elements

Canonical join representation of x = unique antichain $A =: \mathbf{cjr}(x)$ such that :

$$\begin{cases} \bigvee A = x, \\ A \text{ inclusion minimal,} \\ A \text{ order minimal.} \end{cases}$$



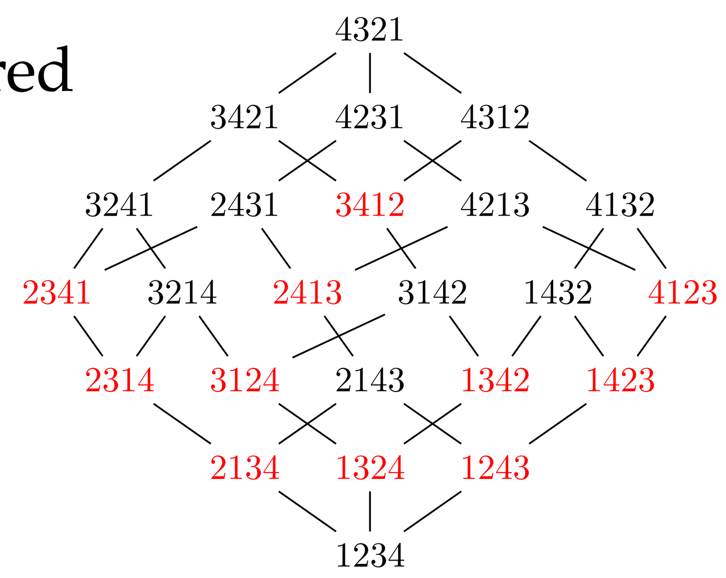
Join-semidistributive lattice L = all elements have a canonical join representation.

$\{$***Join-irreducible elements***$\} = \mathcal{JI}(L) = \{x \mid \mathbf{cjr}(x) = \{x\}\}$.

Analogous definitions for meets.

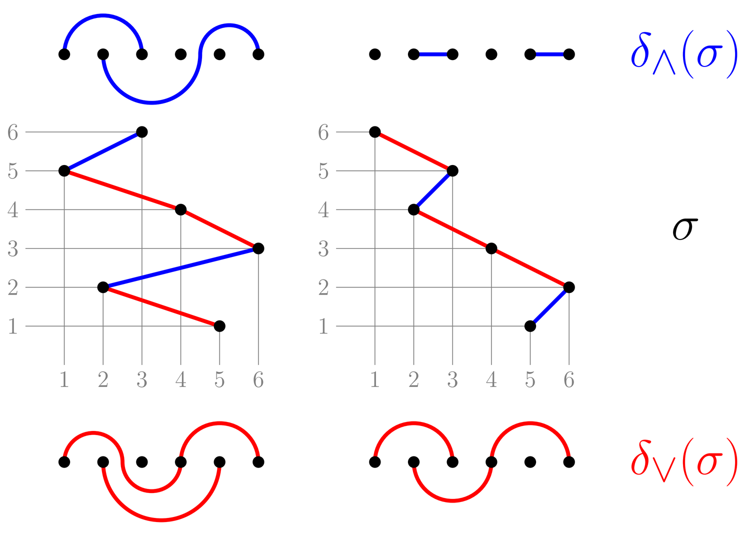
Non-crossing arc diagrams [Rea15]

Weak order = permutations ordered by inclusion of inversion sets.



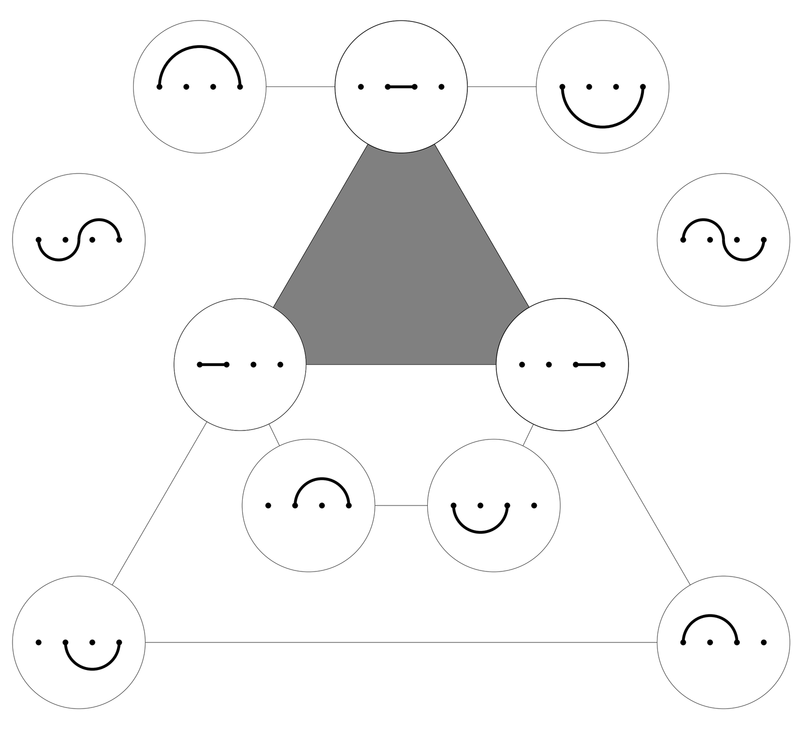
Non-crossing arc diagrams (NCAD) = set of pairwise non-intersecting arcs with no common left or right endpoint.

Theorem. *$\delta_\vee$ and $\delta_\wedge$ are bijections between permutations and NCADs. Moreover,*
$\mathbf{cjr}(\sigma) = \{\delta_\vee^{-1}(\alpha) \mid \alpha \in \delta_\vee(\sigma)\}$,
$\mathbf{cmr}(\sigma) = \{\delta_\wedge^{-1}(\alpha) \mid \alpha \in \delta_\wedge(\sigma)\}$.
In particular, join-irreducible permutations are single arcs.



Canonical join complex [Rea15, Bar19]

$\mathscr{C}_\vee\mathscr{C}(L)$ = simplicial complex on $\mathcal{JI}(L)$ whose faces are the join canonical representations.



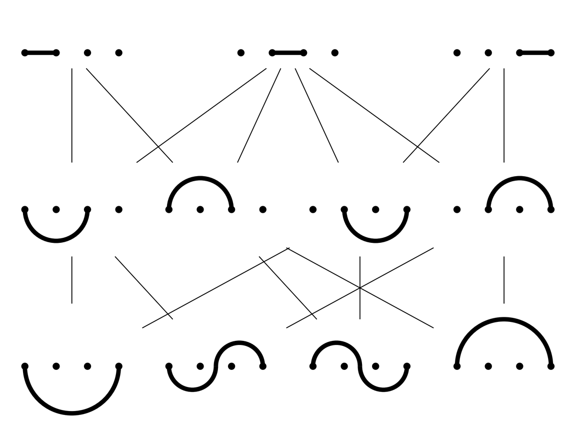
Proposition. *$\mathscr{C}_\vee\mathscr{C}(L)$ is flag. (minimal non-faces are edges)*

Proposition. *For the weak order, $\mathscr{C}_\vee\mathscr{C}$ is (isomorphic to) the non-crossing complex.*

Canonical join complexes of quotients

Congruence = equivalence relation $\equiv$ such that
$x \equiv x'$ and $y \equiv y' \quad \Rightarrow \quad x \vee y \equiv x' \vee y'$ and $x \wedge y \equiv x' \wedge y'$.

Theorem. *A congruence $\equiv$ is determined by the set $\mathcal{I}_\equiv^\vee$ of join-irreducibles uncontracted by $\equiv$. These sets are the ideals of the forcing order.*



Quotient $L/\!\equiv$ = lattice of classes of $\equiv$.

Theorem. *$\mathscr{C}_\vee\mathscr{C}(L/\!\equiv)$ is the subcomplex of $\mathscr{C}_\vee\mathscr{C}(L)$ induced by $\mathcal{I}_\equiv^\vee$.*

References

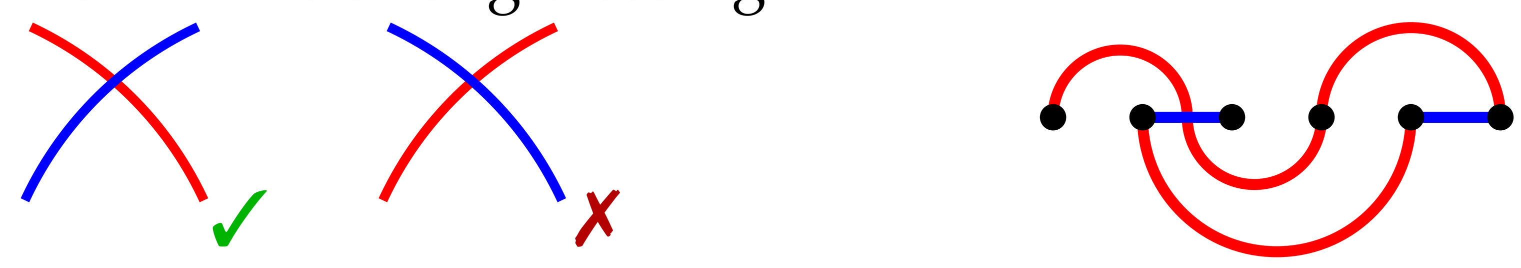
D. A. & V. Pilaud, The canonical complex of the weak order ('22+)
E. Barnard, The canonical join complex ('19)
N. Reading, Noncrossing arc diagrams and canonical join representations ('15)

Canonical representations of intervals

Canonical representation of $[x, y] = \mathbf{cjr}(x) \sqcup \mathbf{cmr}(y)$.

Semi-crossing arc bidiagrams

Semi-crossing arc bidiagram $= D_\vee \sqcup D_\wedge$ the union of two NCADs with no strong crossing.



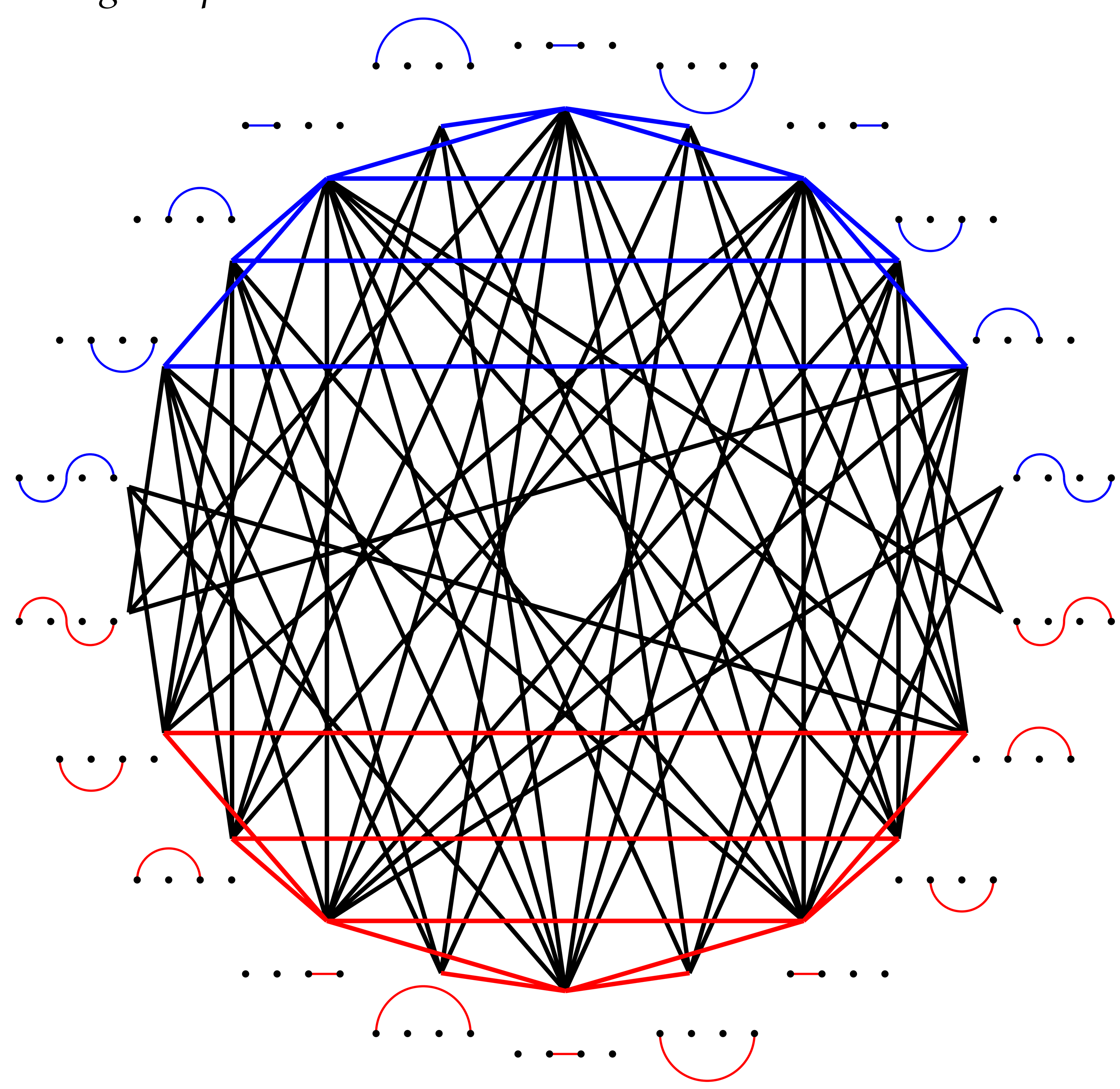
Canonical complex

$\mathscr{C}\mathscr{C}(L)$ = simplicial complex on $\mathcal{JI}(L) \sqcup \mathcal{MI}(L)$ whose faces are the canonical representations.

Proposition. *$\mathscr{C}\mathscr{C}(L)$ is flag.*

Proposition. *$\mathscr{C}\mathscr{C}(L)$ contains $\mathscr{C}_\vee\mathscr{C}(L)$ and $\mathscr{C}_\wedge\mathscr{C}(L)$.*

Proposition. *For the weak order, $\mathscr{C}\mathscr{C}$ is (isomorphic to) the semi-crossing complex.*



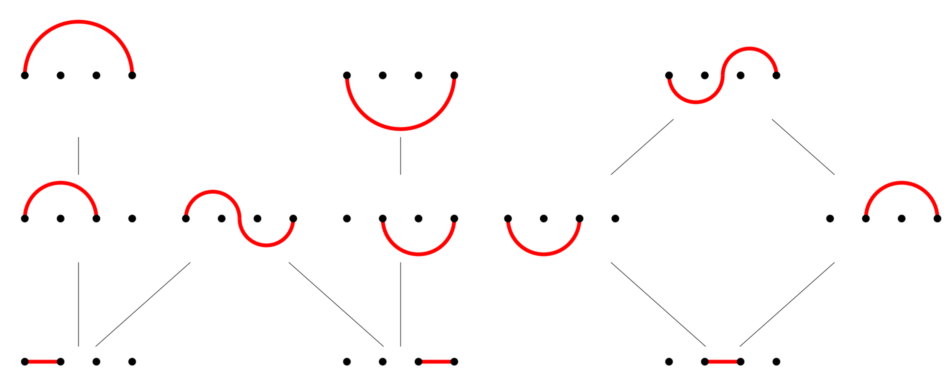
Canonical complexes and quotients

Theorem. *$\mathscr{C}\mathscr{C}(L/\!\equiv)$ is the subcomplex of $\mathscr{C}\mathscr{C}(L)$ induced by $\mathcal{I}_\equiv^\vee$ and $\mathcal{I}_\equiv^\wedge$.*

Kreweras complement

Algorithm. Given $\equiv$ and $\delta_\vee(\pi_\downarrow^\equiv(\sigma))$, find $\delta_\wedge(\pi_\uparrow^\equiv(\sigma))$.

- Take the lower ideal induced by $\delta_\vee(\pi_\downarrow^\equiv(\sigma))$ in the weak order on arcs,
- intersect with $\mathcal{I}_\equiv^\vee$,
- remove fusions,
- take the maximal elements.



Problem. *Generalize to any semidistributive lattice ?*